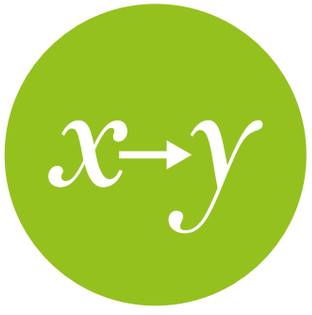
Logistic Regression

Measurement & Evaluation of HCC Systems



Logistic Regression

Today's goal:

Evaluate the effect of multiple variables on a categorical outcome variable

Outline:

- Basic theory: extending regression to logistic regression
- Logistic regression (binary outcome)
- Poisson regression* (count outcome)
- Ordered categorical regression* (Likert scales, etc.)

Extending regression

to logistic regression

A quick aside...

Regression with interaction effect:

$$Y_i = a + b_1X_{1i} + b_2X_{2i} + b_3X_{1i}X_{2i} + e_i$$

You can do this with any X!

Just make sure that your variables are **centered**

Centering a factor:

Assign contrasts that sum to zero

Centering a continuous X:

Subtract the mean

A quick aside...

$Y_i = a + b_1X_{1i} + b_2X_{2i} + b_3X_{1i}X_{2i} + e_i$

Interpretation if X_1 is continuous and X_2 is binary:

b_3 is the additional effect of X_1 in the second group of X_2

b_3 is the additional difference between the two groups of X_2 with each 1 point increase in X_1

Interpretation if both are continuous:

b_3 is the additional effect of X_1 with each 1 point increase of X_2 (and vice versa)

Logistic regression

Linear regression:

$$Y_i = a + b_1X_{1i} + b_2X_{2i} + \ldots + b_kX_{ki} + e_i$$

What if Y is binary (0 or 1)?

We can try to predict the **probability** of Y=1: P(Y)

However, this probability is a number between 0 and 1

For linear regression, we want an unbounded linear Y!

Can we find some transformation that allows us to do this?

Yes: $P(Y) = 1 / (1+e^{-U})$

Logistic regression

$P(Y) = 1 / (1+e^{-U})$

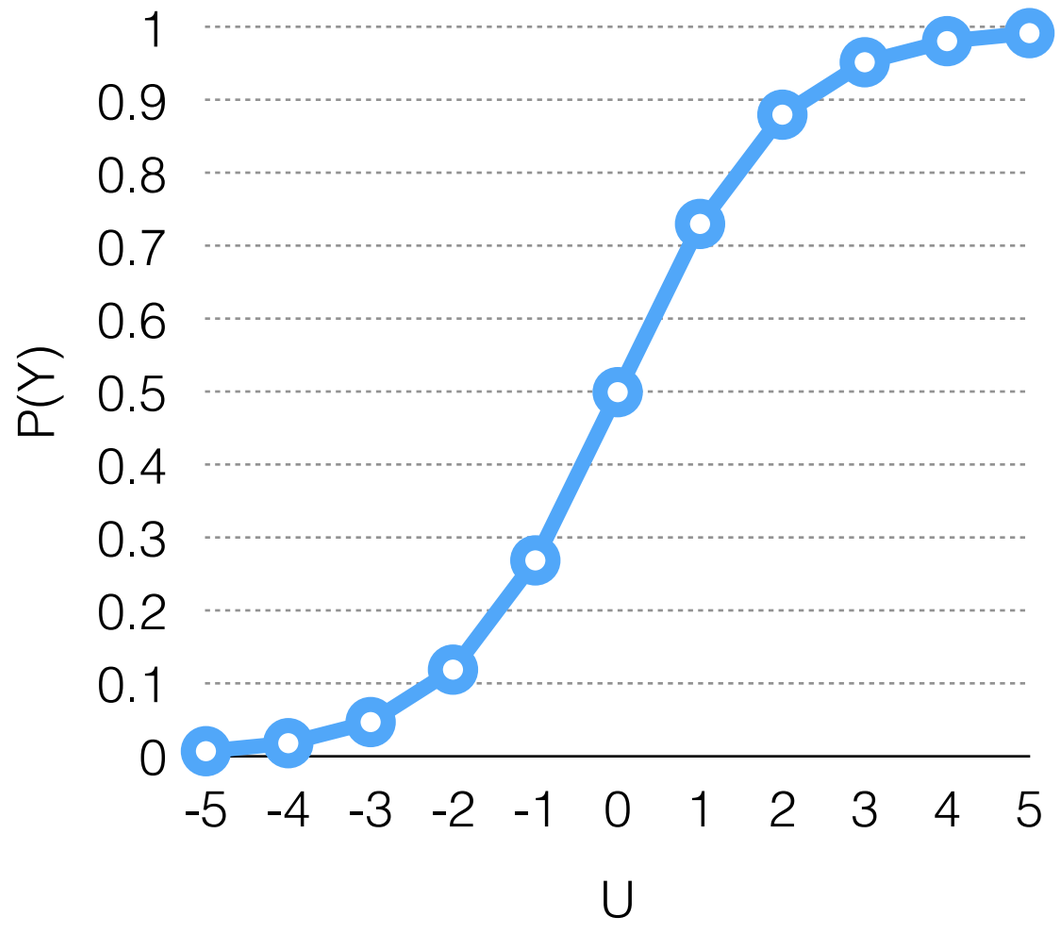
Conversely:

$U = \ln(P(Y)/(1-P(Y)))$

Interpretation:

$P(Y)/(1-P(Y))$ is the **odds** of Y

Therefore, U is the log odds, or **logit** of Y



Logistic regression

Since U is unbounded, we can treat it as our regression outcome:

$$U_i = \ln(P(Y_i)/(1-P(Y_i))) = a + b_1X_{1i} + b_2X_{2i} + \ldots + b_kX_{ki} + e_i$$

We can always transform it back to $P(Y_i)$ if we want to:

$$P(Y_i) = 1 / (1+e^{-(a + b1X1i + b2X2i + \ldots + bkXki + ei)})$$

Log-likelihood

How do we assess the fit of a logistic regression?

We calculate the **log-likelihood**, which is a type of residual

$$\text{Log-likelihood} = \sum(Y_i * \ln(P(Y_i)) + (1 - Y_i) * \ln(1 - P(Y_i)))$$

where Y_i is the observed value, and $P(Y_i)$ is the predicted value

Log-likelihood

Log-likelihood = $\sum(Y_i * \ln(P(Y_i)) + (1-Y_i) * \ln(1-P(Y_i)))$

If $Y_i = 1$, then this simplifies to $\ln(P(Y_i))$

which is zero when the prediction is correct ($P(Y_i)=1$) but gets a large (negative) value if the prediction is incorrect ($P(Y_i)$ is closer to 0)

If $Y_i = 0$, then this simplifies to $\ln(1-P(Y_i))$

which is zero when the prediction is correct ($P(Y_i)=0$) but gets a large (negative) value if the prediction is incorrect ($P(Y_i)$ is closer to 1)

Deviance (–2LL)

A more useful measure is deviance (a.k.a. –2LL)

–2 * log-likelihood

Difference can be used to compare nested models

Likelihood ratio: $\chi^2 = -2LL_{baseline} - -2LL_{new}$

Chi-square distribution with $k_{new} - k_{baseline}$ df

In regression we compared against the mean

In logistic regression we compare against the majority class (either 0 or 1)

Information criteria

Using –2LL to compare non-nested models:

Akaike Information Criterion (AIC):

$$AIC = -2LL + 2k$$

Bayesian Information Criterion (BIC):

$$BIC = -2LL + 2k*\log(N)$$

R^2

We can use –2LL to calculate R^2, but there is some disagreement on how to do this

Hosmer and Lemeshow method:

$$R_L^2 = (-2LL_{baseline} - -2LL_{new}) / -2LL_{baseline}$$

Cox and Snell method:

$$R_{CS}^2 = 1 - \exp((-2LL_{new} - -2LL_{baseline})/N)$$

Nagelkerke method:

$$R_N^2 = R_{CS}^2 / (1 - \exp(2LL_{baseline}/N))$$

Coefficients

In regression, we can test the significance of the b coefficients with a t-test ($t = b/SE_b$)

In logistic regression, this is a z-test

$z = b/SE_b$ (Wald statistic)

The Wald statistic is prone to inflating type II errors, though

Better to just do likelihood ratio model comparisons

Coefficients

How to interpret the b coefficients?

b is the increase in U for each increase of X

b is the increase in ln(P(Y)/(1–P(Y))) for each increase in X

e^b is the ratio of P(Y)/(1–P(Y)) for each increase in X

e^b is the **odds ratio**

Coefficients

Odds ratio examples:

If $e^b > 1$: The odds of Y are e^b times as high for each increase in X

E.g. $e^b = 3$: The odds of Y are 3 times as high for each increase in X

If $e^b < 1$: The odds of Y are $1/e^b$ times as low for each increase in X

E.g. $e^b = .333$: The odds of Y are 3 times as low for each increase in X

Coefficients

If e^b = 1.xx: each 1 pt increase in X leads to a xx% increase in the odds of Y

E.g. e^b = 1.30: The odds of Y are 30% higher for each increase in X

If e^b = 0.xx: each 1pt increase in X leads to a (100-xx)% decrease in the odds of Y

e^b = 0.70: The odds of Y are 30% lower for each increase in X

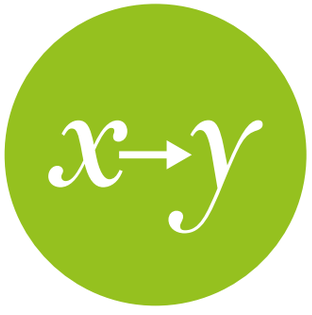
Assumptions

Linearity

In this case, we assume that there is a linear relation between the Xs and the **logit** of Y

Independence

No multicollinearity



Problems

Some times a logistic regression does not converge

You will get weirdly large standard errors

1. You have no or little data for some combinations of Xs

This is especially problematic when Xs are nominal

2. One or a combination of Xs are a perfect predictor of Y

The odds ratios are infinite!

Solution:

Collect more data, or use a simpler model!

Logistic regression

in R

Logistic regression

Dataset “eel.dat”

Effect of a treatment on constipation

Variables:

Cured: whether the patient was cured

Intervention: whether the patient received “No Treatment” or the “Intervention”

Duration: how long the patient had been constipated

Logistic regression

Relevel the Cured variable so that “Not Cured” becomes the baseline:

```
eel$Cured <- relevel(eel$Cured, “Not Cured”)
```

Relevel the Intervention variable so that “No Treatment” becomes the baseline:

```
eel$Intervention <- relevel(eel$Intervention, “No Treatment”)
```

Plotting

Plot of the difference in cured percentage between No Treatment and Intervention, with bootstrapped CI:

```
ggplot(eel, aes(Intervention, as.numeric(Cured == "Cured"))) + stat_summary(fun.y=mean, geom="bar", fill="white", color="black") + stat_summary(fun.data = mean_cl_boot, geom="errorbar", width=0.2) + ylim(0,1)
```

Note:

I'm using as.numeric(Cured == "Cured") to turn this factor into a 0-1 variable...

Plotting

Cured percentage by duration, with bootstrapped CI:

```
ggplot(eel, aes(eel$Duration, as.numeric(Cured ==
"Cured"))) + stat_summary(fun.y=mean, geom="line") +
stat_summary(fun.data=mean_cl_boot, geom = "errorbar",
width=0.2)
```

Split by Intervention:

```
ggplot(eel, aes(eel$Duration, as.numeric(Cured ==
"Cured"), color=Intervention)) + stat_summary(fun.y =
mean, geom="line") + stat_summary(fun.data =
mean_cl_boot, geom = "errorbar", width=0.2)
```

Running a model

Run the model:

```
eel1 <- glm(Cured~Intervention, data=eel, family=binomial)
summary(eel1)
```

This gives us:

- Estimates of the X variable (more on this later)
- Deviance of the baseline model +df
- Deviance of the current model (residual deviance) + df

Model statistics

Model chi-square:

Likelihood ratio: ratio <- eel1$null.deviance – eel1$deviance

Degrees of freedom: df <- eel1$df.null – eel1$df.residual

You can also get these from anova(eel1)

p-value: 1 - pchisq(ratio, df)

R-square:

Hosmer-Lemeshow: ratio / eel1$null.deviance

Cox-Snell: Rcs <- 1-exp(-ratio/113)

Nagelkerke: Rcs / (1-exp(-eel1$null.deviance/113))

Coefficients

```
                           Estimate Std. Error z value Pr(>|z|)
(Intercept)                 -0.2877     0.2700  -1.065  0.28671
InterventionIntervention    1.2287     0.3998   3.074  0.00212 **
```

Calculate percentages:

With no treatment: $P(Y) = 1/(1+e^{-(-0.2877)}) = .429$

With treatment: $P(Y) = 1/(1+e^{-(-.2877+1.2287)}) = .719$

Coefficients

```
                          Estimate Std. Error z value Pr(>|z|)
(Intercept)                -0.2877     0.2700  -1.065  0.28671
InterventionIntervention    1.2287     0.3998   3.074  0.00212 **
```

The intervention has a significant effect

The z-score may be underestimated

What does b = 1.23 mean?

calculate e^b: exp(eel1$coefficients)

The odds of a treated patient being cured are 3.42 higher than those of a patient who is not treated!

Coefficients

Do the same thing for confidence intervals:

exp(confint(eel1))

Note: these are not based on the Wald statistic!

Does not cross 1, therefore, the intervention is significant

Adding Duration

Run the model:

eel2 <- glm(Cured~Intervention+Duration, data=eel, family=binomial)

Interpret the results: summary(eel2)

Duration does not have a significant effect

Deviance very similar to eel1

What is the difference? anova(eel1, eel2)

Significance? 1–pchisq(eel1$deviance–eel2$deviance, eel1$df.residual–eel2$df.residual)

Diagnostics

Diagnostics are largely the same as with linear regression:

You can inspect multicollinearity using VIF

You can get standardized residuals, Cook's distances, leverage and covariance ratios

Diagnostics

Test for linearity of continuous Xs:

Calculate the interaction of the X with its log:

```
eel$logDurationInt <- eel$Duration*log(eel$Duration)
```

Add this to the model:

```
eel3 <- glm(Cured~Intervention+Duration+logDurationInt,
data=eel, family=binomial)
```

If logDurationInt is significant, then there is non-linearity

Reporting

Use a table like Table 8.2 in Field

Report not just b and SE_b, but also the odds ratio (and maybe its confidence interval)

Make sure to report R^2 (Nagelkerke is most accepted), Model χ^2, df, and p-value

If you test multiple models, present the delta R^2 and results of the χ^2 ratio test

Poisson regression

Something that's not in the book!

Poisson regression

Dataset “awards.dat”

Awards won by high school students

Variables:

id: student id

num_awards: number of awards won

prog: type of high school program the student is in

math: the student’s math score

Plotting

Make sure "General" is the baseline type of school:

```
awards$prog <-relevel(awards$prog, ref="General")
```

Histogram by academic program:

```
ggplot(awards,aes(num_awards,fill=prog)) +
geom_histogram(binwidth=0.5, position="dodge")
```

A problem...

Doesn't look very normal!

This is because num_awards is a count variable!

Other examples: # of purchases, # of clicks, time*, price*

Can we find some transformation that makes this work?

Yes: $Y = e^U$

Coefficients

How to interpret the b coefficients?

b is the increase in U for each increase of X

b is the increase in the **log rate** of Y for each increase in X

e^b is the ratio of rate Y for each increase in X

e^b is the **rate ratio**

Why the ratio?

$b = \log(rate_{X+1}) - \log(rate_X) = \log(rate_{X+1} / rate_X)$

therefore, $e^b = rate_{X+1} / rate_X$

Let's try an lm first

Run the model:

alm <- lm(num_awards~prog+math, data=awards)

R^2 = 0.277

Coefficients:

Students in an academic program have 0.48 more awards than students in a general program

For each 1pt increase in math score, the number of awards increases with 0.048

Let's try an lm first

Residuals:

```
awards$lmresid <- rstandard(alm)
awards$lmresid.large <- (awards$lmresid > 1.96 |
awards$lmresid < -1.96)

awards[awards$lmresid.large,]
```

Some residuals are huge (> 3.29)

Let's try a glm

Run the model:

aglm <- glm(num_awards~prog+math, data=awards, family=poisson)

R-square:

R^2_{hl}: (aglm$null.deviance–aglm$deviance) / aglm$null.deviance

R^2_{cs}: 1-exp((aglm$deviance–aglm$null.deviance)/200)

R^2_{n}: Rcs / (1-exp(-aglm$null.deviance/200))

Better model fit!

Let's try a glm

Coefficients: exp(aglm$coefficients)

Students in an academic program have 2.96 times as many awards than students in a general program

For each 1pt increase in math score, the number of awards increases with 7.27%

Do the same thing for confidence intervals:

exp(confint(aglm))

Note: these are not based on the Wald statistic!

Significant when they do not cross 1

Let's try a glm

Residuals:

awards$glmresid <- rstandard(aglm)

awards$glmresid.large <- (awards$glmresid > 1.96 |
awards$glmresid < -1.96)

awards[awards$glmresid.large,]

No huge residuals!

Ordered logistic

Also not in the book!

Ordered logistic

Dataset “consequences.dat”

Consideration of future consequences questionnaire

Variables:

age, gender: participant’s age and gender

Q3: answer to the question “I only act to satisfy immediate concerns, figuring the future will take care of itself.”

answer categories: 1=extremely uncharacteristic, 2=somewhat uncharacteristic, 3=uncertain, 4=somewhat characteristic, 5=extremely characteristic

A problem...

This is ordinal, not interval!

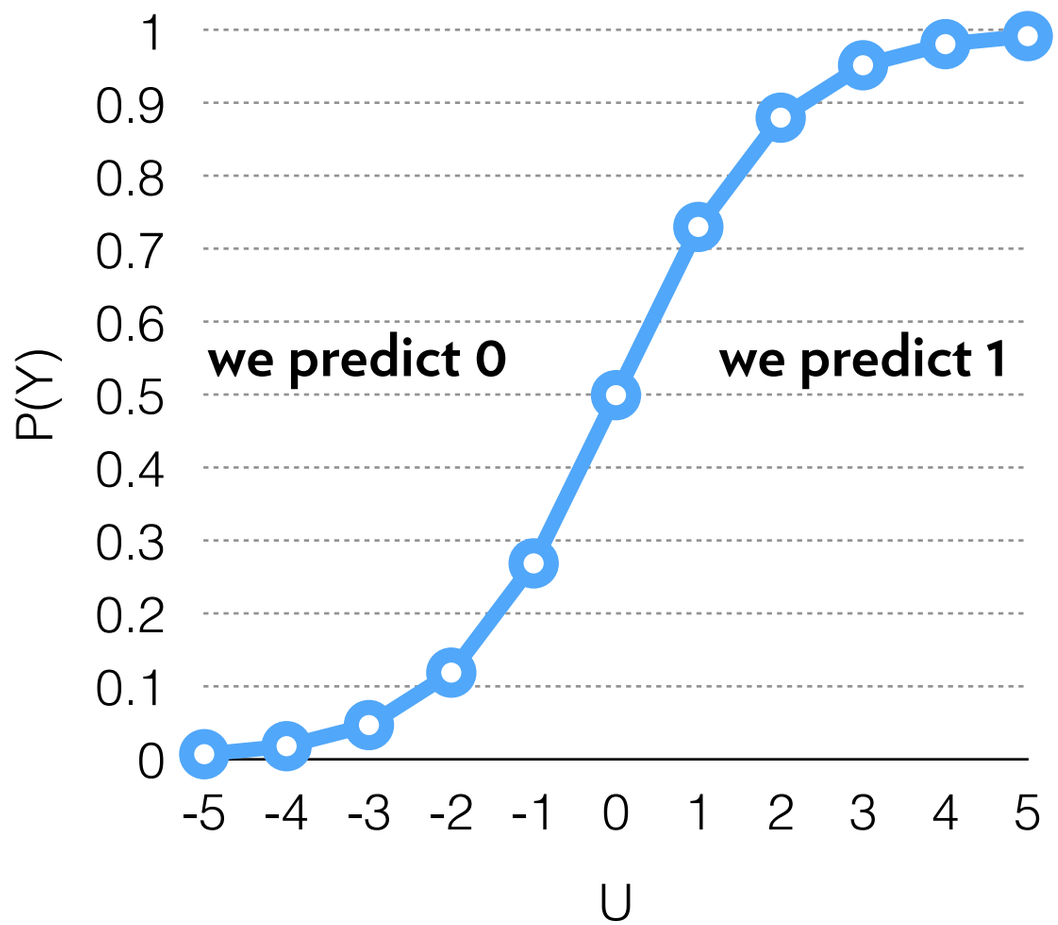
> Is the difference between “extremely uncharacteristic” and “somewhat uncharacteristic” the same as the difference between “uncertain” and “somewhat characteristic”?

Also, not very normally distributed!

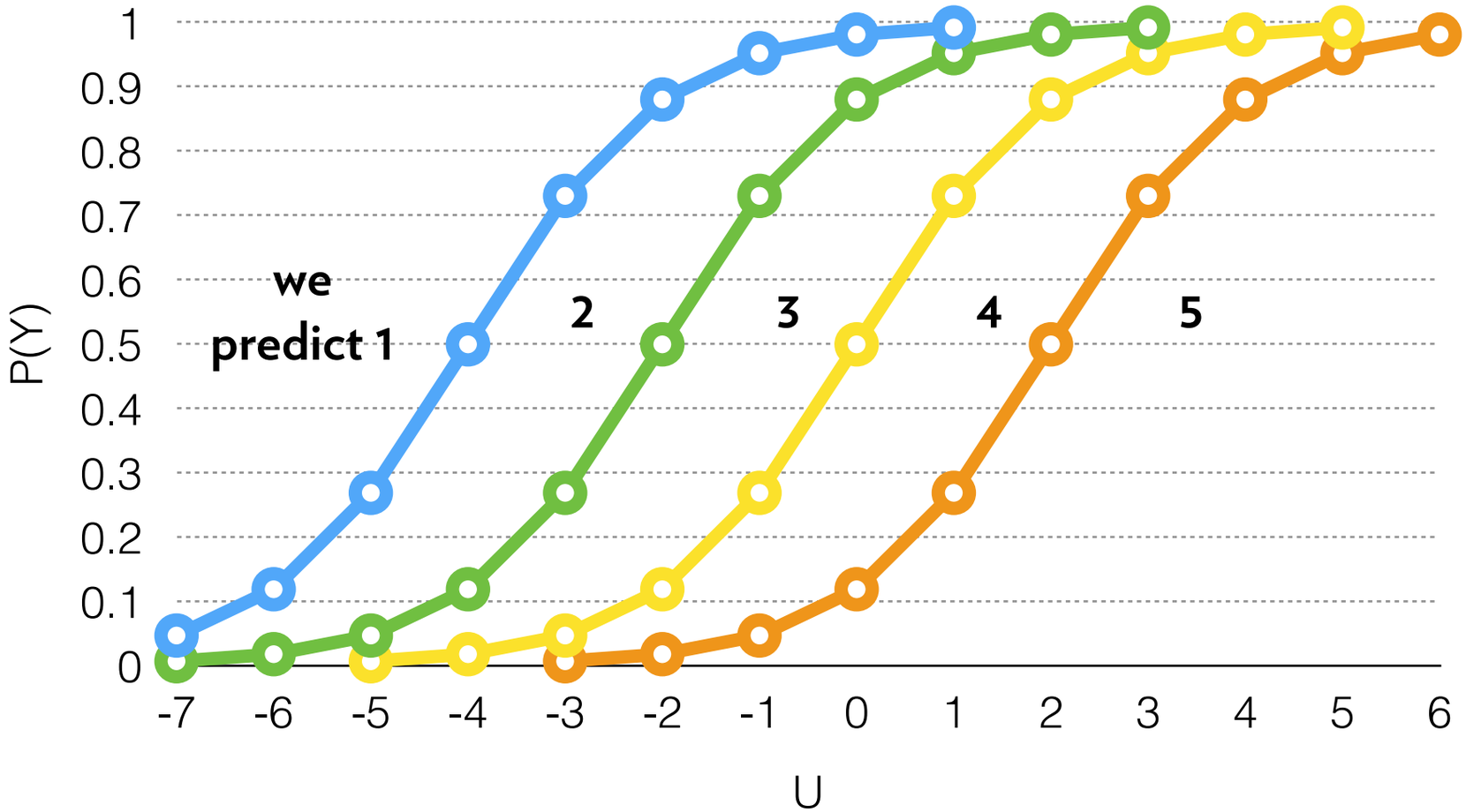
```
ggplot(consequences,aes(Q3))+stat_bin(binwidth=1)
```

How can we solve these problems?

Logistic regression



Ordered logistic



Coefficients

The model estimates intercepts for each threshold

1|2, 2|3, 3|4, 4|5

These thresholds are the **log odds** of any person having **at least** this value

How to interpret the b coefficients?

e^b is the **odds ratio** for a 1pt increase in X

e.g. if the odds ratio is 1.40, then the odds of a certain value or higher increases by 40% if X is 1 higher

Let's try an lm first

Run the model:

clm <- lm(Q3~gender+age, data=consequences)

R^2 = 0.030

Coefficients:

- Females score higher on satisfying immediate concerns (not significant)
- Older individuals score lower on satisfying immediate concerns (not significant)

Let's try a polr

Run the model:

```
cplm <- polr(factor(Q3)~gender+age, data=consequences,
Hess=T)
```

Run a null model:

```
cplm.null <- polr(factor(Q3)~1,data=consequences,
Hess=T)
```

Let's try a polr

R-square:

R^2_{hl}: (cplm.null$deviance–cplm$deviance) / cplm.null$deviance

R^2_{cs}: 1-exp((cplm$deviance–cplm.null$deviance)/199)

R^2_{n}: Rcs / (1-exp(-cplm.null$deviance/199))

The latter two suggest a better model fit!

Let's try a polr

Coefficients: exp(cplm$coefficients)

Females have a 70% higher likelihood to rate higher on satisfying immediate concerns

For each 1 year increase in age, the likelihood to rate higher on satisfying immediate concerns decreases by 2.09%

Do the same thing for confidence intervals:

exp(confint(cplm))

Note: these are not based on the Wald statistic!

Significant when they do not cross 1

Robust methods

Also not in the book!

Robust methods

Bootstrapping works the same as linear regression

Alternative method: sandwich estimator of SE (package: "sandwich") — this also works for regular lm!

```
cov.aglm <- vcovHC(aglm, type="HC0")
std.err <- sqrt(diag(cov.aglm))
pval <- 2 * pnorm(abs(coef(aglm)/std.err), lower.tail=F)
LL <- coef(aglm) - 1.96 * std.err
UL <- coef(aglm) + 1.96 * std.err
```

“It is the mark of a truly intelligent person
to be moved by statistics.”

T H A N K S !

George Bernard Shaw